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Technical Notes

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Heat Transfer with a Step in Surface Temperature

M. Sandoval and C. Treviño*
Universidad Nacional Autónoma de México, 04510 México, D. F., México

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Nomenclature

C, C_1	=	numerical constants
h	=	thickness of the plate
$K(x, x')$	=	integral kernel
k	=	thermal conductivity of the fluid
k_w	=	thermal conductivity of the leading portion of the plate, $x \leq x_0$
Nu	=	Nusselt number
Nu^*	=	upstream Nusselt number
Pr	=	Prandtl number, ν/α
Q_s	=	upstream heat flux per unit length at the interface position, $x = x_0$
q_w	=	heat flux to the fluid per unit surface of the plate
R_0	=	Reynolds number, $U_\infty x_0/\nu$
r	=	inner coordinate, $(1 - s)/\varepsilon$
s	=	nondimensional longitudinal coordinate, x/x_0
T_w	=	temperature at the plate upper surface
T_{wl}	=	temperature at the leading edge of the plate ($x = 0$)
T_0	=	temperature of the second portion of the plate, for $x \geq x_0$
T_∞	=	far-stream temperature of the fluid
U_∞	=	freestream fluid velocity
u	=	longitudinal fluid velocity component in the x direction
x, y	=	Cartesian coordinates with the origin at the upper surface of the leading edge of the plate
x_p	=	thermal penetration length
x_0	=	interface position of the two plate materials
z	=	nondimensional longitudinal coordinate, $(3/4)^{5/18} s^{3/4}/\varepsilon^{5/6}$
α	=	thermal diffusivity of the fluid
δ	=	viscous boundary-layer thickness
δ_T	=	thermal boundary-layer thickness
ε	=	ratio of the thermal penetration length to x_0
ν	=	kinematic fluid viscosity
θ_w	=	nondimensional temperature at the upper surface of the plate

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*Facultad de Ciencias, Departamento de Física; ctrev@servidor.unam.mx (Corresponding Author).

I. Introduction

THIS Note deals with the classical problem of the heat transfer in a laminar boundary-layer flow of an incompressible fluid along a flat plate, with a step in surface temperature occurring in a finite length from the leading edge of the plate [1,2]. The case of a compressible fluid is not addressed here [3–5]. Because of the fact that when using constant thermal properties of the fluid (thermal conductivity, density, and specific heat), the energy equation is linear and independent of the momentum and the mass conservation equations, there is an integral representation of the resulting heat flux to the fluid as a function of the temperature distribution of the plate. A special case is that when the temperature experiences an abrupt change from the temperature of that corresponding to the fresh fluid T_∞ to any given value $T_0 \neq T_\infty$. Lighthill [6] obtained a similar solution by assuming that the thermal boundary-layer thickness is very small compared with the viscous boundary-layer thickness, which, in fact, gives an excellent approximation if the thicknesses are of the same order. The heat flux in this case shows a singularity at the position at which the surface temperature jumps. To have an step change in surface temperature at some longitudinal length from the leading edge of the plate, x_0, a different material with different thermal properties has to be employed. Basically, a material with negligibly small thermal conductivity (perfect insulator) is needed to cover the initial length of the plate. For a material with finite thermal properties, a small but finite heat flux is transferred upstream from the interface of the two sections of the flat plate, thus perturbing the heat flux to the fluid and destroying its singular behavior at the interface position. To take into account a material with finite thermal properties, it is necessary to solve the coupled convection and solid conduction governing equations known as the conjugate heat transfer problem [7–10]. The objective of this work is to evaluate the implications of using plate materials with finite thermal properties.

II. Analysis

The physical model analyzed is shown in Fig. 1. A thin flat plate of length L and thickness $2h$ is placed parallel to a forced flow of an incompressible fluid with freestream velocity U_∞, temperature T_∞, and conductivity k. The plate is composed of two materials with different thermal properties. The first material with finite thermal conductivity k_w is located from the leading edge by a length x_0, followed by the second material with extremely large thermal conductivity, which is assumed to have a uniform temperature T_0. The Cartesian coordinates are x in the longitudinal direction and y in the transverse direction, with the origin at the upper left corner of the plate. An order-of-magnitude estimate allows the heat transfer at the interface between both portions of the plate:

$$Q_s \sim \frac{k_w(T_0 - T_\infty)h}{x_p} \sim \frac{k(T_0 - T_\infty)x_p}{\delta_T} \tag{1}$$

where x_p denotes the characteristic thermal penetration length measured from the interface position (see Fig. 1), which is initially assumed to be very small compared with x_0, and δ_T is the characteristic thickness of the thermal boundary layer. From the balance of the convective and diffusive terms in the energy equation in the fluid, the following relationship can be obtained:

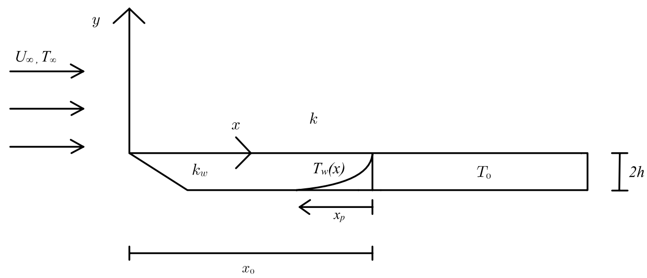

Fig. 1 Schematics of the studied heat transfer problem.

$$\frac{u(T_0 - T_\infty)}{x_p} \sim \frac{\alpha(T_0 - T_\infty)}{\delta_T^2} \tag{2}$$

where u corresponds to the local fluid velocity $u \sim U_\infty \delta_T/\delta$ (with δ denoting the viscous boundary-layer thickness), and α is the thermal diffusivity of the fluid. With $x_p \ll x_0$, then $\delta \sim x_0/R_0^{1/2}$, where R_0 is the Reynolds number based on x_0, $R_0 = U_\infty x_0/\nu$ (with ν being the kinematic viscosity of the fluid). Using relationships (1) and (2), the ratio of the thermal penetration length to x_0 is obtained by

$$\varepsilon = \frac{x_p}{x_0} = \frac{C_1}{Pr^{1/5} R_0^{3/10}} \left(\frac{h}{x_0}\frac{k_w}{k}\right)^{3/5} \tag{3}$$

where $C_1 = (3/4)^{1/3}[4/f''(0)]^{1/5} = 1.495$ is a constant that is included for convenience. Here, $f''(0) = 0.332$ corresponds to the second derivative of the Blasius function evaluated at the surface of the plate and Pr is the Prandtl number of the fluid, $Pr = \nu/\alpha$. Parameter ε arises as the most important parameter to describe the transition from a step in surface temperature ($\varepsilon = 0$) to the case of an uniform plate temperature ($\varepsilon \to \infty$). The restriction is that $x_p \ll x_0$ was used only to obtain a clear definition of parameter ε by Eq. (3). That is, for values of $\varepsilon \ll 1$, this parameter relates the ratio of the thermal penetration length to x_0. For values of ε of order unity, the thermal perturbation reaches the leading edge of the plate.

The nondimensional boundary-layer governing equations for an incompressible fluid along a flat plate with variable temperature are given by [1]

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \tag{4}$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2} f \frac{\partial\theta}{\partial\eta} = s \frac{df}{d\eta}\frac{\partial\theta}{\partial s} \tag{5}$$

where

$$\theta(s,\eta) = \frac{T - T_\infty}{T_0 - T_\infty}; \qquad s = \frac{x}{x_0}; \qquad f = \frac{1}{\sqrt{\nu U_\infty x}} \int_0^y u \, dy'; \qquad \eta = y\sqrt{\frac{U_\infty}{\nu x}} \tag{6}$$

Equations (4) and (5) have to be solved with the following boundary conditions:

$$f(0) = \left.\frac{df}{d\eta}\right|_{\eta=0} = \left.\frac{df}{d\eta}\right|_{\eta\to\infty} - 1 = \theta(x,0) - \theta_w(x) = \theta(x,\infty) = 0 \tag{7}$$

where $\theta_w(x)$ is the nondimensional temperature at the interface plate fluid to be obtained after solving the energy (Laplace) equation for the plates. An integral solution to Eqs. (4–7) was obtained by Lighthill [6] for a given surface temperature as

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0} = -\left(\frac{f''(0)Pr}{4}\right)^{1/3}\left[\theta_{\mathrm{wl}} + \int_{\theta_{\mathrm{wl}}}^{\theta_w(s)} K(s,s')\, d\theta_w'(s')\right] \tag{8}$$

where $K(s,s')$ is the integral kernel defined by $K(s,s') = [1 - (s'/s)^{3/4}]^{-1/3}$, and θ_{wl} corresponds to the nondimensional temperature at the leading edge of the plate, $\theta_{wl} = \theta_w(0)$. However, for thin plates ($h/x_0 \ll 1$), the thermally thin approximation can be introduced and the plate temperature variations in the transverse direction can be neglected. It is to be noticed that using the thermally thin wall approximation, the temperature variation, but not its gradient in the transverse direction, is neglected. In this sense, integrating the Laplace equation in the transverse direction and applying the boundary conditions (continuity of temperature and heat flux at the interface plate fluid, together with the symmetry condition of adiabaticity at the lower plate surface $y = -h$), the following equation for the nondimensional temperature of the plate is obtained:

$$\frac{d^2\theta_w}{ds^2} = \frac{(3/4)^{5/9}}{\varepsilon^{5/3}\sqrt{s}}\left[\theta_{wl} + \int_{\theta_{wl}}^{\theta_w(s)} K(s,s')\, d\theta_w\right] \tag{9}$$

to be solved with the following boundary conditions:

$$\frac{d\theta_w}{ds}(0) = 0; \qquad \theta_w(1) = 1 \tag{10}$$

The first boundary condition (10) assumes that all of the heat is transferred to the fluid by the upper surface of the plate or that the heat flux through the leading edge can be neglected, compared with that through the upper surface of the plate, which, in fact, is a good approximation for enough thin plates ($h/x_0 \ll 1$). The limiting solutions to Eqs. (9) and (10) for small and large values of ε compared with unity are $\theta_w = H(s-1)$ for $\varepsilon = 0$ and $\theta_w = 1$ for $\varepsilon \to \infty$. Here, $H(t)$ corresponds to the Heaviside step functions $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t > 0$. The nondimensional heat flux to the fluid is then given by

$$Nu = \frac{q_w x_0}{k(T_0 - T_\infty)}\left(\frac{4}{f''(0)}\right)^{1/3}\frac{1}{Pr^{1/3}R_0^{1/2}} = \frac{1}{\sqrt{s}}\left[\theta_{wl} + \int_{\theta_{wl}}^{\theta_w(s)} K(s,s')H(1-s')\, d\theta_w\right] \tag{11}$$

For $\varepsilon = 0$, $Nu = H(s-1)[1-(1/s)^{3/4}]^{-1/3}s^{-1/2}$, which is the well-known solution for a step in surface temperature, and $Nu = s^{-1/2}$ for $\varepsilon \to \infty$ corresponds to the solution for a constant plate temperature. The asymptotic solution for small but finite values of ε compared with unity (that is, $x_p \ll x_0$) can be obtained by introducing the appropriate inner coordinate $r = (1-s)/\varepsilon$. The governing equation (9) transforms to

$$\frac{d^2\theta_w(r)}{dr^2} = \int_0^{\theta_w} \frac{d\theta_w}{\sqrt[3]{r' - r}} \tag{12}$$

with the following boundary conditions:

$$\theta_w(0) = 1; \qquad \theta_w(\infty) = 0 \tag{13}$$

The solution to Eqs. (12) and (13) can be readily obtained as

$$\theta_w(r) = \exp(-Cr) \tag{14}$$

where C is a constant given by $C = [\Gamma(2/3)]^{3/5}$, where $\Gamma(2/3)$ is the gamma function, $\Gamma(2/3) = \int_0^\infty t^{-1/3}\exp(-t)\, dt = 1.3541$, and $C \simeq 1.1995$.. The Nusselt number in this regime is given by

$$Nu = \left(\frac{4}{3\varepsilon}\right)^{1/3} C^2 \exp(-Cr) \quad \text{for } r \geq 0 \qquad (s < 1) \tag{15}$$

and

$$Nu = \left(\frac{4C}{3\varepsilon}\right)^{1/3} \exp(-Cr)\Gamma(-r/C, 2/3) \quad \text{for } r < 0 \quad (s > 1) \tag{16}$$

where $\Gamma(-r/C, 2/3)$ corresponds to the upper incomplete gamma function, given by

$$\Gamma(-r/C, 2/3) = \int_{-r/C}^{\infty} t^{-1/3} \exp(-t)\,\mathrm{d}t$$

On the other hand, for large values of ε compared with unity, the process tends to a constant uniform temperature of the whole plate. From Eq. (9), we can deduce the behavior for large values of ε compared with unity. For finite large values of ε, the plate temperature can be obtained and is given by

$$\frac{\theta_w}{\theta_{wl}} = 1 + \frac{1.1364}{\varepsilon^{5/3}} s^{3/2} + \frac{0.29056}{\varepsilon^{10/3}} s^3 + \mathcal{O}(\varepsilon^{-5}) \quad \text{for } \varepsilon \gg 1 \tag{17}$$

with θ_{wl} given by

$$\theta_{wl} = 1 - \frac{1.1364}{\varepsilon^{5/3}} + \frac{1.0008}{\varepsilon^{10/3}} + \mathcal{O}(\varepsilon^{-5}) \quad \text{for } \varepsilon \gg 1 \tag{18}$$

Another parameter of interest is the upstream heat flux per unit length from the interface between both plate materials , Q_s, at $x = x_0$. In nondimensional form, the appropriate Nusselt number Nu^* is then

$$Nu^* = \frac{Q_s x_0}{h k_w T_0 - T_\infty} = \left.\frac{\partial \theta_w}{\partial s}\right|_{s=1} \tag{19}$$

The asymptotic behaviors for small and large values of ε compared with unity are

$$Nu^* \sim \frac{C}{\varepsilon} \quad \text{for } \varepsilon \to 0 \quad \text{and} \quad Nu^* \sim \frac{1.7046}{\varepsilon^{5/3}} + \frac{0.87169}{\varepsilon^{10/3}} + (\varepsilon^{-5}) \quad \text{for } \varepsilon \to \infty \tag{20}$$

III. Results

For the numerical calculations, it is convenient to introduce a longitudinal coordinate as $z = (3/4)^{5/18} s^{3/4}/\varepsilon^{5/6}$. With this new coordinate, the governing equation (9) transforms to

$$3z\frac{\mathrm{d}^2\theta_w}{\mathrm{d}z^2} - \frac{\mathrm{d}\theta_w}{\mathrm{d}z} = \frac{16}{3} z \left[\theta_{wl} + \int_0^z \frac{\mathrm{d}\theta'_w/\mathrm{d}z'}{(1 - z'/z)^{1/3}}\,\mathrm{d}z'\right] \tag{21}$$

to be solved with the boundary conditions $\mathrm{d}\theta_w/\mathrm{d}z|_0 = 0$ and $\theta_w(z_f) = 1$, where $z_f = (3/4)^{5/18}/\varepsilon^{5/6}$. Equation (21) is a parameter-free equation, for which the parameter ε was shifted to the boundary condition. This equation can be solved as an initial-value equation, with the initial conditions $\mathrm{d}\theta_w/\mathrm{d}z|_0 = 0$ and $\theta_w(0) = \theta_{wl}(\varepsilon)$. We specify the value of θ_{wl} and obtain z_f as the position at which θ_w crosses the value of unity. Once we know the value of z_f, we can obtain the value of the parameter ε as $\varepsilon = (4/3)^{1/3} z_f^{6/5}$. Figure 2 shows the nondimensional temperature profiles as a function of the normalized coordinate s for different values of the parameter ε by solving Eq. (21) with the corresponding boundary conditions. The relation between the nondimensional temperature at the leading edge of the plate ($s = 0$) and parameter ε is shown in Fig. 3, in which the asymptotic behaviors for small and large values of ε compared with unity are also plotted. For small values of ε, an extremely good approximation is by using the approximate relationship $\theta_{wl} = \exp(-1/\varepsilon)$ for $\varepsilon \to 0$. The inset of Fig. 3 shows this relationship and that it gives good results for values up to 0.05. However, the asymptotic behavior for large values of ε given by relationship (18) only reproduces the solution for relatively large values of ε around 10. Figure 4 shows the numerical results of the Nusselt number Nu as a function of the normalized coordinate s for different values of parameter ε. The relationships for $\varepsilon = 0$ and $\varepsilon \to \infty$ are also plotted. The singularity at $s = 1$ disappears for small but finite values of ε, producing a peak in the Nusselt number at this position, with decreasing values as the value of ε increases. A singularity now appears at $s = 0$. Finally, there is a smooth transition to the solution obtained for constant plate temperature ($\varepsilon \to \infty$) as the value of ε further increases. The peak at $s = 1$ disappears for values of $\varepsilon \gg 1$. The reduced Nusselt number $Nu\varepsilon^{1/3}$ for small values of ε was obtained from Eqs. (15) and (16) and is shown in

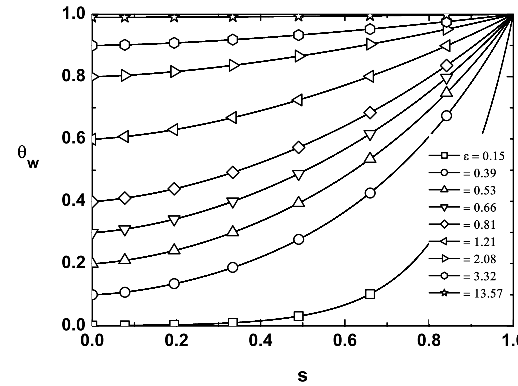

Fig. 2 Nondimensional temperature profiles for different values of parameter ε.

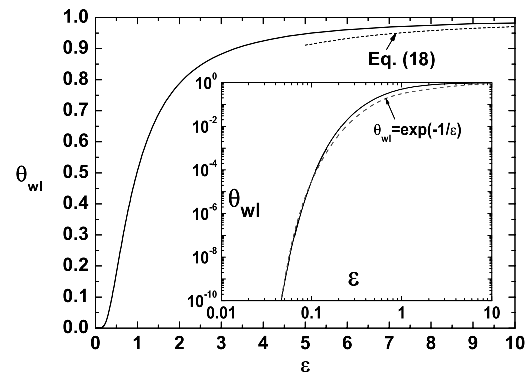

Fig. 3 Nondimensional temperature at the leading edge as a function of ε. The asymptotic relationships for small and large values of ε compared with unity are also plotted.

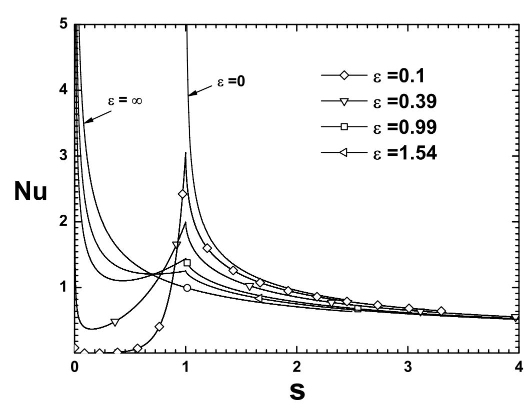

Fig. 4 Nusselt number Nu as a function of the normalized coordinate s for different values of parameter ε.

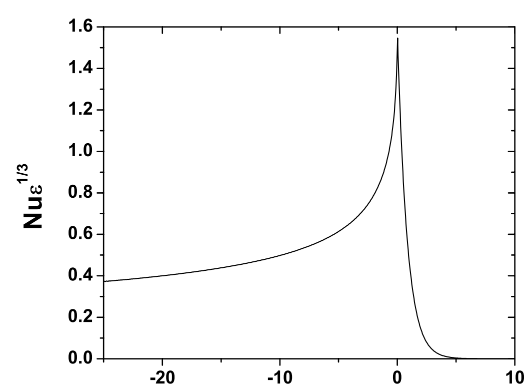

Fig. 5 Reduced Nusselt number $Nu\varepsilon^{1/3}$ as a function of the inner coordinate r.

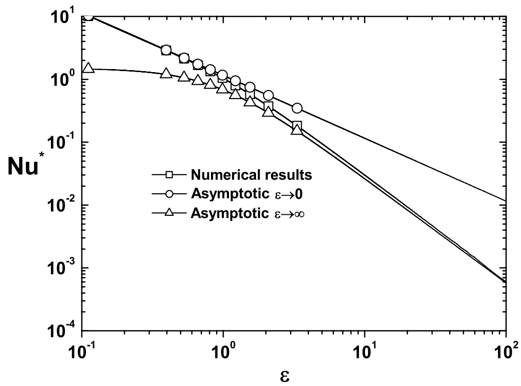

Fig. 6 Nusselt number *Nu as a function of parameter *ε*. Numerical and asymptotic relationships for small and large values of *ε* compared with unity.**

Fig. 5 as a function of r. The singular behavior for $\varepsilon \to 0$ can be well reproduced in this figure. Positive values of r correspond to values of $s < 1$, with $s = 1 - \varepsilon r$. The maximum value of the Nusselt number at $s = 1$ is then $(4/3)^{1/3} C^2/\varepsilon^{1/3} \simeq 1.5836/\varepsilon^{1/3}$ for $\varepsilon \to 0$. Finally, Fig. 6 shows Nu^* as a function of ε obtained numerically from Eq. (21). The asymptotic behaviors for small and large values of ε compared with unity, given by relations (20) are also plotted in this figure. The asymptotic solution for small values of ε reproduces Nu^* well up to values of ε of order unity.

IV. Conclusions

In this work, the heat transfer due to a step in surface temperature was revised and the implications due to the thermal properties of the preceding plate material were deduced. A thermal penetration length x_p was obtained and its ratio to the initially unheated distance x_0 was designated as the important parameter that affects the heat transfer characteristics. The transition from a step in surface temperature to the constant temperature plate was obtained as parameter $\varepsilon = x_p/x_0$ increased from zero to ∞. The nondimensional heat flux or Nusselt number was obtained as a function of this parameter ε, showing that the singular behavior moves from $x = x_0$ to $x = 0$ for finite values of ε.

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